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Riemannian geometry and stability of ideal quantum gases

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Abstract. It is shown that the stability of ideal quantum gases can be measured by means of the Riemann scalar curvature R of the parameter space. The components of the metric tensor were assumed to be the second moments of energy and the number of particle fluctuations. As a result, R is a function of the second and third moments of those quantities. For bosons R is positive and increases monotonically from zero at the classical limit to positive infinity in the condensation region. A system is less stable if R is bigger and vice versa. For fermions R is negative and this means that Fermi gases are more stable than the ideal Bose and ideal classical systems.

1. Introduction

Geometrical methods have always played an important role in thermodynamics. They not only facilitate the analysis of systems in thermodynamics of equilibrium states, but also give a better understanding and deeper insight into the mathematical structure of the theory. Recently it has been shown that the empirical laws of phenomenological thermodynamics may be expressed in a mathematically rigorous and concise way if one uses the language of contact geometry. This approach to problems of equilibrium thermodynamics was originated by Hermann [1] and developed in [2]. Another approach to the geometry of thermodynamics is based on the concept of the distance between thermodynamic states. On a purely phenomenological level, it was initiated by Weinhold [3] who introduced a sort of Riemannian metric into the space of thermodynamic parameters by means of a scalar product of some reference vectors, tangent to the manifold of thermodynamic states. Many authors later discussed the physical consequences which resulted from the Weinhold construction.

Gilmore [4] proposed another definition of the metric tensor and argued that there was no natural measure of distance in thermodynamics but there was a natural measure of curvature. Ruppeiner [5] included the theory of fluctuations in the axioms of thermodynamics and showed that this leads to a reasonable Riemannian metric on a manifold of equilibrium states. Elements of his metric tensor were represented by the second moments of fluctuations of some parameters. He also proposed to connect the Riemann curvature of the thermodynamic manifold with interparticle effective strength of interaction in the system. However, it turned out that the two metrics introduced by Weinhold and Ruppeiner are conformally equivalent [6, 7].

A statistical approach to the metrisation problems in thermodynamics was initiated by Ingarden [8] who defined the metric tensor by means of the relative entropy. These ideas have been further developed in [9, 10]. In our previous paper [11] we presented a unified statistical and phenomenological approach to the metrisation problem of a

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space of thermodynamic parameters. We gave a general formula of transformations of the metric tensor under Legendre transformations.

In the present paper we investigate the case of ideal Bose and Fermi gases of indistinguishable particles. It was known [9] that for an ideal classical gas the scalar curvature R is always zero. As we will see, this is not the case for ideal quantum gases. Although the interparticle interactions are absent (ideal gases), the effects of quantum statistics cause the behaviour of ideal quantum gases to be quite different from that of a classical ideal gas. In the quantum case there exist spatial correlations between various particles of the system. They originate from the symmetry properties of the wavefunctions for indistinguishable particles. The quantum effects, i.e. those spatial correlations, become quite significant at low temperatures and (or) for a high density of particles.

2. Metric geometry of the parameter space

Let us consider a quantum mechanical system described by an equilibrium density matrix ρ,

$$\rho = Z^{-1}(\beta)\exp(-\beta^i F_i) \qquad i=1,2,\ldots,r \tag{2.1}$$

(in all formulae we assume summation over repeated indices) which depends on r self-adjoint and linearly independent operators $F_1, F_2, \ldots, F_r$ and on r classical real parameters $\beta^1, \beta^2, \ldots, \beta^r$. $Z(\beta)$ is the standard partition function or the normalisation factor, i.e.

$$Z(\beta) = \mathrm{Tr}\exp(-\beta^i F_i). \tag{2.2}$$

Let us denote by m_i the statistical mean values of the quantities represented by F_i:

$$m_i = \langle F_i \rangle \equiv \mathrm{Tr}(\rho F_i). \tag{2.3}$$

Physically, F_i represent quantities which may fluctuate freely and the only constraints are those imposed by (2.3), which means that the numerical values of m_i are fixed, if only ρ is fixed. On the other hand, the parameters β^i do not fluctuate and they characterise an environment (bath) surrounding our system. In such a way changes in the β^i, caused by changes of the state of the environment, generate changes of the density matrix ρ and numerical values of m_i. In this sense (2.1) may be treated as an r-parameter family of density matrices. On the parameter space we may define a Riemannian structure by means of the following formula [9-11]:

$$\mathrm{d}l^2 = \langle(\mathrm{d}\ln\rho)^2\rangle = \left\{\frac{1}{2}\left\langle\frac{\partial\ln\rho}{\partial\beta^i}\frac{\partial\ln\rho}{\partial\beta^j}\right\rangle + \frac{1}{2}\left\langle\frac{\partial\ln\rho}{\partial\beta^j}\frac{\partial\ln\rho}{\partial\beta^i}\right\rangle\right\}\mathrm{d}\beta^i\,\mathrm{d}\beta^j. \tag{2.4}$$

If the operators F_i commute the above formula takes the form

$$\begin{aligned}\mathrm{d}l^2 = \langle(\mathrm{d}\ln\rho)^2\rangle &= \left\langle\frac{\partial\ln\rho}{\partial\beta^i}\frac{\partial\ln\rho}{\partial\beta^j}\right\rangle\mathrm{d}\beta^i\,\mathrm{d}\beta^j \\ &= -\left\langle\frac{\partial^2\ln\rho}{\partial\beta^i\,\partial\beta^j}\right\rangle\mathrm{d}\beta^i\,\mathrm{d}\beta^j = \frac{\partial^2\ln Z(\beta)}{\partial\beta^i\,\partial\beta^j}\mathrm{d}\beta^i\,\mathrm{d}\beta^j.\end{aligned} \tag{2.5}$$

The above formula may be derived [8] from the relative information (relative entropy). Let us consider two close statistical states $\rho=\rho(\beta)$ and $\sigma=\rho(\beta+\mathrm{d}\beta)$. According to

[8] we define the information distance $I(\rho|\sigma)$ between these two states (information gain) as

$$I(\rho|\sigma)=\mathrm{Tr}[\rho(\ln\rho-\ln\sigma)]\geq 0. \tag{2.6}$$

Next we expand (2.6) into a power series in the neighbourhood of β up the second-order terms in $\mathrm{d}\beta^i$. In order to do this one has to use the well known formula for the parameter differentiation of an exponential operator

$$\frac{\mathrm{d}\exp(A(\mu))}{\mathrm{d}\mu}=\int_0^1 \mathrm{d}\lambda\,\exp[(1-\lambda)A]\frac{\mathrm{d}A}{\mathrm{d}\mu}\exp(\lambda A)=\int_0^1 \mathrm{d}\lambda\,\exp(\lambda A)\frac{\mathrm{d}A}{\mathrm{d}\mu}\exp[(1-\lambda)A]. \tag{2.7}$$

One can easily show that

$$I(\rho|\rho)=0 \qquad \frac{\partial I}{\partial\beta^i}=0 \qquad i=1,2,\ldots,r \tag{2.8}$$

while for ρ given by (2.1) we obtain

$$\frac{1}{2}\frac{\partial^2 I}{\partial\beta^i\,\partial\beta^j}=\int_0^1 \mathrm{d}\lambda\,\mathrm{Tr}\left(\rho\exp(\lambda\beta^l F_l)\frac{\partial\ln\rho}{\partial\beta^i}\exp(-\lambda\beta^l F_l)\frac{\partial\ln\rho}{\partial\beta^j}\right). \tag{2.9}$$

This may be further written in the form

$$\frac{1}{2}\frac{\partial^2 I}{\partial\beta^i\,\partial\beta^j}=\frac{1}{2}\frac{\partial^2\ln Z}{\partial\beta^i\,\partial\beta^j}=-\frac{1}{2}\left\langle\frac{\partial^2\ln\rho}{\partial\beta^i\,\partial\beta^j}\right\rangle. \tag{2.10}$$

Finally the local square distance in the parameter space is given by

$$\mathrm{d}l^2=2I(\rho(\beta+\mathrm{d}\beta)|\rho(\beta))=\frac{\partial^2\ln Z(\beta)}{\partial\beta^i\,\partial\beta^j}\mathrm{d}\beta^i\,\mathrm{d}\beta^j. \tag{2.11}$$

If the operators F_i commute this formula is the same as (2.4). In the opposite case, the formula (2.11) differs from (2.4). Because the metric tensor in (2.11) is expressed in terms of the second derivatives of the potential function $\ln Z(\beta)$, this definition of the metric tensor $g_{ij}=\partial^2\ln Z/\partial\beta^i\partial\beta^j$ seems better.

Statistically the components of the metric tensor g_{ij} can be expressed as

$$g_{ij}=\int_0^1 \mathrm{d}\lambda\,\mathrm{Tr}[\rho\exp(\lambda\beta^l F_l)(F_i-\langle F_i\rangle)\exp(-\lambda\beta^l F_l)(F_j-\langle F_j\rangle)]. \tag{2.12}$$

If $i=j$ we have second moments of F_i. In the case $i\neq j$ we get the covariances of F_i and F_j. In the case of commuting operators

$$g_{ij}=\langle(F_i-\langle F_i\rangle)(F_j-\langle F_j\rangle)\rangle. \tag{2.13}$$

3. The geometrical structure of the parameter space for the systems described by the grand canonical distribution

The simplest thermodynamical systems are those with two degrees of freedom, i.e. with $r=2$. To each degree of freedom there corresponds a stochastic quantity (here represented by F_i) which may fluctuate and is conjugate to the statistical temperature (here β^i) which characterises the external conditions. If one chooses for F_1 and F_2 the Hamiltonian operator $\hat{H}$ and the operator of the number of particles $\hat{N}$, respectively, then, by referring to phenomenological thermodynamics, we gather that $\beta^1 \equiv \beta = (kT)^{-1}$ and $\beta^2 \equiv \gamma = -\mu(kT)^{-1}$, where k is the Boltzmann constant, T is the absolute temperature and μ is the chemical potential. The corresponding density matrix

$$\rho = Z^{-1}(\beta, \gamma) \exp(-\beta\hat{H} - \gamma\hat{N}) \tag{3.1}$$

is called the grand canonical (or T-μ) distribution. The partition function $Z(\beta, \gamma)$ is formally equal to

$$Z(\beta, \gamma) = \mathrm{Tr} \exp(-\beta\hat{H} - \gamma\hat{N}) \tag{3.2}$$

but physically it may be expressed by means of the Kramers potential q, $q = k \ln Z$, which in turn is equal to PV/T [12, 13], where P and V are the pressure and volume of the system. Thus we have

$$Z = \exp(q/k) = \exp(PV/kT) = \exp(\alpha V) \qquad \alpha = P/kT. \tag{3.3}$$

The dependence of Z on β and γ means therefore that P is a function of β and γ. Due to (3.3) we have

$$\rho = \exp(-\alpha V - \beta\hat{H} - \gamma\hat{N}) \tag{3.4}$$

and

$$\mathrm{d} \ln \rho = -V\,\mathrm{d}\alpha - \hat{H}\,\mathrm{d}\beta - \hat{N}\,\mathrm{d}\gamma. \tag{3.5}$$

We have to notice, however, that despite the symmetric appearance of V, $\hat{H}$ and $\hat{N}$ in (3.5), their roles are conceptually different. The operators $\hat{H}$ and $\hat{N}$ represent here stochastic quantities whereas the volume V is a fixed classical quantity (not an operator). By means of (3.5) we go over to phenomenological thermodynamics by defining the 1-form θ,

$$\theta := \langle -\mathrm{d} \ln \rho \rangle = V\,\mathrm{d}\alpha + U\,\mathrm{d}\beta + N\,\mathrm{d}\gamma \tag{3.6}$$

where $U = \langle H \rangle$ and N are the mean values of the energy and the number of particles. For a spatially homogeneous system, it is convenient to take

$$\theta' = \theta/V = \mathrm{d}\alpha + u\,\mathrm{d}\beta + n\,\mathrm{d}\gamma \tag{3.7}$$

where $u = U/V$ and $n = N/V$ are densities of energy and particles. It is easy to see that $\langle \mathrm{d} \ln \rho \rangle$ is equal to zero if the density matrix ρ is normalised. The normalisation of ρ thermodynamically means that α is a function of β and γ.

If one treated (temporarily) α, β, γ, u and n as independent variables then the 1-form θ' would define a contact structure on five-dimensional thermodynamical phase space [6] with local coordinates α, β, γ, u and n. In our case the states which can be realised by a given thermodynamical system constitute only a two-dimensional submanifold Σ, called the thermodynamic surface in the thermodynamic phase space. According to the first and second laws of thermodynamics, this surface is given by the following exterior differential equation:

$$\theta' = 0. \tag{3.8}$$

One of the possible solutions of (3.8) has the form

$$\alpha = \alpha(\beta, \gamma) \qquad u = -\partial\alpha/\partial\beta \qquad n = -\partial\alpha/\partial\gamma. \tag{3.9}$$

We have mentioned already, that ρ is normalised, i.e. $\mathrm{Tr}\,\rho = 1$. Due to (3.7) and (3.9), α is called a potential function with respect to β and γ. In the following we will identify the space of parameters β and γ with the thermodynamic surface Σ given by (3.9). Consequently, Σ becomes a Riemannian manifold with the metric tensor (2.10).

If one now notices that due to (3.3)

$$g_{ij}(\beta) = \frac{\partial^2 \ln Z(\beta)}{\partial \beta^i\, \partial \beta^j} = V \frac{\partial^2 \alpha}{\partial \beta^i\, \partial \beta^j} \qquad i, j = 1, 2 \tag{3.10}$$

then it is clear that, in order to calculate the components of the metric tensor g_{ij}, we can use the apparatus of either statistical or phenomenological thermodynamics. Of course, the physical interpretation of the components of g_{ij} is much deeper if we use both methods. Due to (2.12) in the case of commuting operators and (3.2), we have

$$\begin{aligned} g_{11} &= g_{\beta\beta} = \langle (\hat{H} - \langle \hat{H} \rangle)^2 \rangle \\ g_{12} &= g_{\beta\gamma} = \langle (\hat{H} - \langle \hat{H} \rangle)(\hat{N} - \langle \hat{N} \rangle) \rangle \\ g_{22} &= g_{\gamma\gamma} = \langle (\hat{N} - \langle \hat{N} \rangle)^2 \rangle. \end{aligned} \tag{3.11}$$

The statistical interpretation of the components of g is, therefore, very nice and appealing. They simply describe the square correlations of the original stochastic variables. So, they characterise the fluctuations of the quantities which correspond to these variables. On the other hand, the phenomenological interpretation of g_{ij} may be inferred from (3.9) and (3.10) or from

$$g_{ij}(\beta) = \frac{\partial^2 \ln Z(\beta)}{\partial \beta^i\, \partial \beta^j} = -\frac{\partial m_i}{\partial \beta^j} = -\frac{\partial m_j}{\partial \beta^i} \tag{3.12}$$

because $-\partial \ln Z(\beta)/\partial \beta^i = m_i$. Thus we have

$$\begin{aligned} g_{11} &= -\left(\frac{\partial U}{\partial \beta}\right)_\gamma = kT^2 \left(\frac{\partial U}{\partial T}\right)_\gamma \\ g_{12} &= g_{21} = -\left(\frac{\partial U}{\partial \gamma}\right)_\beta = kT \left(\frac{\partial U}{\partial \mu}\right)_T = kT^2 \left(\frac{\partial N}{\partial T}\right)_\mu \\ g_{22} &= -\left(\frac{\partial N}{\partial V}\right)_\beta = kT \left(\frac{\partial N}{\partial \mu}\right)_T = kTN^2 V^{-1} \chi_T \end{aligned} \tag{3.13}$$

where $\chi_T = -V^{-1}(\partial V/\partial P)_{N,T}$ is the isothermal compressibility, and the last equality for g_{22} may be found in [13, section 4.6].

4. Geometrical structure in the case of ideal Bose and Fermi gases

We will now apply the general formalism of the Riemannian geometry developed in the two preceding sections to ideal quantum gases. From (3.9) we see that all the thermodynamic information about a system is contained in the relation $\alpha = \alpha(\beta, \gamma)$. According to (3.10) the knowledge of this relation is also sufficient to evaluate all geometrical quantities which we will need in the following. Unfortunately, this relation, called the fundamental relation, is known only in a few special cases.

The grand canonical distribution applied to an ideal non-relativistic gas of spinless bosons of mass m leads to the following two equations [12, 13]:

$$\ln Z = PV/kT = \lambda^{-3} V \bar{g}_{5/2}(\eta) \tag{4.1}$$

$$n = \lambda^{-3} \bar{g}_{3/2}(\eta) \tag{4.2}$$

where $\lambda = h/(2\pi mkT)^{1/2}$ is the mean thermal wavelength of the particle, h is the Planck constant and

$$\bar{g}_l(\eta) = \frac{1}{\Gamma(l)} \int_0^\infty \frac{x^{l-1}\,\mathrm{d}x}{\eta^{-1}\mathrm{e}^x - 1}. \tag{4.3}$$

$\Gamma(l)$ denotes here the Gamma function and we have used the standard symbol for the fugacity

$$\eta = \mathrm{e}^{-\gamma} = \mathrm{e}^{\mu/kT}. \tag{4.4}$$

In the case of arbitrary integer spin the formulae (4.1) and (4.2) must be multiplied by the degree of degeneracy. These formulae are crucial for our further considerations. All details which lead to them may be found, e.g., in [12, 13] and in other textbooks of statistical physics.

For bosons, $\mu \leqslant 0$, and therefore η varies from 0 to 1. We must stress, however, that although the relation (4.1) is in principle exact, the relation (4.2) holds only for the high-temperature region in which the number of particles in the ground state is negligible. It means that those values of temperature for which we observe the Bose–Einstein condensation are excluded from our considerations.

It may be easily checked that for η satisfying $0 < \eta < 1$, $\bar{g}_l(\eta)$ may be expanded in powers of η, and one gets

$$\bar{g}_l(\eta) = \sum_{j=1}^{\infty} \eta^j j^{-l} \tag{4.5}$$

which, in the interesting range of η, converges for $l > 1$. We will also need $\bar{g}_l(\eta)$ for $l = \frac{1}{2}$ and $l = -\frac{1}{2}$. In these cases the sequence $\bar{g}_l(\eta)$ converges for $0 < \eta < 1$ but diverges for $\eta = 1$. From (4.3) or (4.5) one gets an important relation

$$\bar{g}_{l-1}(\eta) = \eta \frac{\partial \bar{g}_l(\eta)}{\partial \eta} \tag{4.6}$$

which holds for each l.

According to (4.1) and (4.2) the components of the metric tensor are as follows:

$$\begin{aligned} g_{11} &= \frac{\partial^2 \ln Z}{\partial \beta^2} = \tfrac{15}{4}\lambda^{-3} V \beta^{-2} \bar{g}_{5/2}(\eta) = \tfrac{15}{4}\beta^{-2} PV/kT \\ g_{12} &= \frac{\partial^2 \ln Z}{\partial \beta\, \partial \gamma} = \tfrac{3}{2}\lambda^{-3} V \beta^{-1} \bar{g}_{3/2}(\eta) = \tfrac{3}{2} NkT \\ g_{22} &= \frac{\partial^2 \ln Z}{\partial \gamma^2} = \lambda^{-3} V \bar{g}_{1/2}(\eta). \end{aligned} \tag{4.7}$$

The determinant of g expressed through the function $\bar{g}_l(\eta)$ has the form

$$\det g = \tfrac{3}{4}\lambda^{-6} V^2 \beta^{-2} [5\bar{g}_{5/2}(\eta)\bar{g}_{1/2}(\eta) - 3\bar{g}^2_{3/2}(\eta)]. \tag{4.8}$$

We see that this metric degenerates for $\eta = 0$. This causes no trouble since the states with $\eta = 0$ are unphysical. If we set $\bar{g}_l(\eta) = \eta$ then we obtain the classical case. For small but not negligible values of η (the classical or high-temperature limit) one may assume $\bar{g}_l(\eta) \approx \eta$ and then $\det g \approx \frac{3}{2}\lambda^{-6}V^2\beta^{-2}\eta^2$ is positive. On the other hand, $\det g$ tends to positive infinity if $\eta \to 1$ as $\bar{g}_l(1)$ diverges for $l \leqslant 1$.

The Christoffel symbols [14]

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial \beta^k} + \frac{\partial g_{ik}}{\partial \beta^j} - \frac{\partial g_{jk}}{\partial \beta^i}\right) \tag{4.9}$$

reduce in our case to

$$\Gamma_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial \beta^k} = \frac{1}{2}\frac{\partial^3 \ln Z}{\partial \beta^i \, \partial \beta^j \, \partial \beta^k}. \tag{4.10}$$

In turn, the Riemann curvature tensor

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial \beta^i \, \partial \beta_l} - \frac{\partial g_{ik}}{\partial \beta^j \, \partial \beta^l} + \frac{\partial g_{il}}{\partial \beta_j \, \partial \beta^k} - \frac{\partial g_{jl}}{\partial \beta^i \, \partial \beta^k}\right) + g^{mn}(\Gamma_{mil}\Gamma_{njk} - \Gamma_{mik}\Gamma_{njl}) \tag{4.11}$$

reduces to

$$R_{ijkl} = g^{mn}(\Gamma_{mil}\Gamma_{njk} - \Gamma_{mik}\Gamma_{njl}) \tag{4.12}$$

because the first bracket in (4.11) disappears due to the special choice of the metric tensor $g_{ik} = \partial^2 \ln Z/\partial \beta^i \partial \beta^k$.

We consider here systems with two thermodynamic degrees of freedom and therefore the dimension of the thermodynamical surface Σ (given by $\theta = 0$ or by (3.9)) is equal to 2. On the two-dimensional Riemann manifold there is only one non-vanishing component of R_{ijkl}, namely R_{1212}. Thus the scalar curvature

$$R = g^{mn} R^i_{nim} \tag{4.13}$$

is given by a very simple formula [14]

$$R = \frac{2}{\det g} R_{1212}. \tag{4.14}$$

It is interesting to note that all components of Γ_{ijk}, R_{ijkl} and R itself are expressed through the second and third derivatives of $\ln Z$. Because of the special form of the metric tensor the formula for R may be presented as

$$R = \frac{2}{(\det g)^2}\begin{vmatrix} g_{11} & g_{22} & g_{12} \\ \partial g_{11}/\partial \beta^1 & \partial g_{22}/\partial \beta^1 & \partial g_{12}/\partial \beta^1 \\ \partial g_{11}/\partial \beta^2 & \partial g_{22}/\partial \beta^2 & \partial g_{12}/\partial \beta^2 \end{vmatrix}. \tag{4.15}$$

From the formulae below, one can also see that, from the statistical point of view, all these quantities are expressed through the second and third moments and correlations of $\hat{H}$ and $\hat{N}$. In fact, due to (3.1), (3.2) and (4.1), we have

$$\begin{aligned}
\frac{\partial^3 \ln Z}{\partial \beta^3} &= -\langle(\hat{H} - \langle\hat{H}\rangle)^3\rangle = -\tfrac{105}{8}\lambda^{-3} V\beta^{-3}\bar{g}_{5/4}(\eta) \\
\frac{\partial^3 \ln Z}{\partial \gamma \, \partial \beta^2} &= -\langle(\hat{H} - \langle\hat{H}\rangle)^2(\hat{N} - \langle\hat{N}\rangle)\rangle = -\tfrac{15}{4}\lambda^{-3} V\bar{g}_{3/2}(\eta)\beta^{-2} \\
\frac{\partial^3 \ln Z}{\partial \beta \, \partial \gamma^2} &= -\langle(\hat{H} - \langle\hat{H}\rangle)(\hat{N} - \langle\hat{N}\rangle)^2\rangle = -\tfrac{3}{2}\lambda^{-3} V\beta^{-1}\bar{g}_{1/2}(\eta) \\
\frac{\partial^3 \ln Z}{\partial \gamma^3} &= -\langle(\hat{N} - \langle\hat{N}\rangle^3\rangle = -\lambda^3 V\bar{g}_{-1/2}(\eta).
\end{aligned} \tag{4.16}$$

This procedure may not be continued because the fourth derivatives of $\ln Z$ cannot be expressed through the fourth moments. In fact, they can be expressed through the moments of fourth and lower order and are fourth cumulants. Finally, from (4.7), (4.8), (4.15) and (4.16) we get

$$R = 20\lambda^3 V^{-1} \frac{\bar{g}_{3/2}^2(\eta)\bar{g}_{1/2}(\eta) - 2\bar{g}_{5/2}(\eta)\bar{g}_{1/2}^2(\eta) + \bar{g}_{5/2}(\eta)\bar{g}_{3/2}(\eta)\bar{g}_{-1/2}(\eta)}{[5\bar{g}_{5/2}(\eta)\bar{g}_{1/2}(\eta) - 3\bar{g}_{3/2}^2(\eta)]^2}. \tag{4.17}$$

For an ideal Fermi gas of particles of spin s the whole procedure may be repeated if one replaces $\bar{g}_l(\eta)$ by $f_l(\eta)$ in all formulae, where

$$f_l(\eta) = \frac{1}{\Gamma(l)} \int_0^\infty \frac{x^{l-1}\,dx}{\eta^{-1}\,e^x + 1}. \tag{4.18}$$

For $\eta < 1$, $f_l(\eta)$ may be also expanded in powers of η with the result

$$f_l(\eta) = \sum_{j=1}^{\infty} (-1)^{j-1} \eta^j j^{-l}. \tag{4.19}$$

For fermions, the chemical potential μ may be either negative or positive, $-\infty < \mu < \infty$, and therefore

$$0 < \eta < \infty. \tag{4.20}$$

As a result, the power expansion (4.19) does not hold for the whole range of η. For $\eta > 1$ one has to take the exact formula (4.18) or use another expansion in powers of $(\ln \eta)^{-1}$ [12]. In this paper, however, we are confined only to $0 < \eta < 1$ and consequently (4.19) is quite satisfactory. As we said, the formula (4.17) for scalar curvature holds also for fermions, but we will replace $\bar{g}_l(\eta)$ by $f_l(\eta)$:

$$R = 20\lambda^3 V^{-1}(2s+1)^{-1} \frac{f_{3/2}^2(\eta)f_{1/2}(\eta) - 2f_{5/2}(\eta)f_{1/2}^2(\eta) + f_{5/2}(\eta)f_{3/2}(\eta)f_{1/2}(\eta)}{[5f_{5/2}(\eta)f_{1/2}(\eta) - 3f_{3/2}^2(\eta)]^2}. \tag{4.21}$$

The counterpart of (4.2) for fermions is more reliable than (4.2) itself for bosons. Neglecting the ground state for fermions is quite unimportant as the eigenvalues of the particle number operator in this state are only 0 and 1.

The formula (4.17) for the scalar curvature R is very complicated. As a matter of fact, we are not even able to say whether R is positive or negative for various values of β and γ (or λ and η). It is immediately seen that, in the classical case, $\bar{g}_l(\eta) = f_l(\eta) = \eta$, the scalar curvature is 0. This result agrees with the earlier calculation performed for the ideal classical gas in [9].

In table 1 we have collected some numerical values of R which were computed with a very high accuracy. For bosons R is given in units of $20\lambda^3 V^{-1}$, while for fermions it is given in units of $20\lambda^3 V^{-1}(2s+1)^{-1}$. Computations were performed for $1/kT = $ constant, i.e. for an isotherm. This is sufficient because at present we are interested only in a quantative behaviour of R for the ideal Bose (Fermi) gas. All isotherms look qualitatively the same.

From table 1 we see that for bosons R is always positive and monotonically increases as η tends to 1. The scalar curvature R diverges to positive infinity. For fermions R is negative in the interval (0, 1).

Table 1. Scalar curvature R for chosen values of the fugacity η for bosons (in units of $20\lambda^3 V^{-1}$) and fermions (in units of $20\lambda^3 V^{-1}(2s+1)^{-1}$).

	Scalar curvature R	
η	bosons	fermions
0.100	0.4539×10^{-1}	-0.7334×10^{-1}
0.300	0.4852×10^{-1}	-0.1805
0.500	0.5337×10^{-1}	-0.4916
0.700	0.6245×10^{-1}	-0.2226×10^{1}
0.900	0.9187×10^{-1}	-0.6608×10^{4}
0.910	0.9563×10^{-1}	-0.2909×10^{4}
0.920	0.1001	-0.4131×10^{3}
0.930	0.1054	-0.1575×10^{3}
0.940	0.1121	-0.8268×10^{2}
0.950	0.1207	-0.5095×10^{2}
0.960	0.1323	-0.3459×10^{2}
0.970	0.1493	-0.2505×10^{2}
0.980	0.1778	-0.1909×10^{2}
0.990	0.2423	-0.1493×10^{2}

The parameter V appearing in all formulae is not essential. It may be omitted by redefining the metric tensor as

$$\bar{g}_{ij}=\frac{1}{V}\frac{\partial^2 \ln Z}{\partial\beta^i\,\partial\beta^j}\qquad \beta^1=\beta\qquad \beta^2=\gamma. \tag{4.22}$$

So the curvature R does not depend on the volume of the system and is a function of second and third moments per unit volume.

5. Concluding remarks

It is experimentally well established that fluctuations in single-phase systems are thermodynamically negligible and, therefore, such systems are relatively stable. Fluctuations become very important in multiphase systems, especially in the vicinity of the critical points. As a result, in the closest vicinity of the critical points, systems become extremely unstable. The scalar curvature R depends on the second and third moments of fluctuations and, therefore, we propose to interpret R as a measure of global fluctuations in the system caused by interactions (in our case quasi-interactions). In this sense R may be treated as a measure of the stability of the system: the bigger R, the less stable is the system. In the classical limit (small η) the bosonic gas is far away from the region in which the Bose–Einstein condensation occurs [12, 13] and is relatively stable. As $\eta\to 1$ the system is closer to the condensation region and is less stable. This condensation may be treated formally as a sort of a phase transition, although it is conceptually different from the well known gas–liquid or liquid–solid transition. A divergence of R to positive infinity for $\eta\to 1$ corresponds to this experimentally confirmed condensation.

Traditionally, departure from the equilibrium mean values (fluctuations) of some quantities are described in terms of the second moments, i.e. by components of the metric tensor in our geometrical formalism. Phenomenologically, this is equivalent to taking the second derivatives of thermodynamic functions. In this paper, we proposed to also take into account the third moments which, in a natural way, enter the expression for the scalar curvature R. Phenomenologically, this means taking the third derivatives of thermodynamic functions, which is not the case in the standard thermodynamics.

For the ideal Bose gases the fluctuations are bigger (positive spatial correlations due to the statistical effect of attraction of particles) than those for the classical ideal gas and hence R is positive.

On the contrary, for ideal Fermi gases the fluctuations are smaller, (negative spatial correlations, due to repulsion of particles) than those for the classical ideal gas and so R is negative. If the stability of the ideal classical gas may be called normal, then that of bosons is ultranormal and that of fermions infranormal in agreement with the Pauli principle.

It must be stressed, however, that this concerns only that part of fluctuations which results from the interparticle interactions (for classical gases) and from the quantum effects for ideal quantum gases.

It seems that for real Bose and Fermi systems similar conclusions may be true.

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